Comparison of crude Monte Carlo, DOE, and worst-case distance methods

1. Explanation of methods

Circuit performances like gain, slew rate, noise figure, delay, etc., vary due to random variation of process variables like tox, Vth, etc. The probability distribution of the process variables is given, and there are different methods to estimate statistics of the circuit performances, like quantiles, mean values, standard deviation. Each performance has a specification bound, for example: „gain > 80dB“. The percentage of circuits that fulfill this specification is the partial yield of „gain“.

In realistic analog circuits, the number of performances to consider is about 10 or less. The number of statistical variables can be very large (100-1000) if mismatch is to be considered, because then each transistor needs two variables.

In the following, an example performance „f“ is considered together with one example process parameter „s“. Figure 1 shows a graph of f(s). The mean value of s is mu. The performance f decreases monotonously over s and violates its lower specification bound for a certain value s=swc.

In crude Monte Carlo, f is simulated for a random sample of size N. Due to the gaussian distribution of s, the density is highest around s=mu.

The estimated yield is the percentage of samples with s≤swc. In Figure 1 therefore, the yield estimate is 100%, although the true yield is certainly smaller, maybe only 97%. This is a statistical error that goes away for large sample sizes N.

+ The slight non-linearity of f(s) is considered correctly,
- but each individual sample causes simulation cost.
+ All estimates of mean values, standard deviation, and yield, have zero bias, i.e. they approach their true value for the limit N → ∞.
- For realistic values of N (like N=1000), there is some statistical error in the estimates of mean, standard deviation and yield, of f.
+ The statistical error is independent of the dimensionality of s.

![Abbildung 1: Crude Monte Carlo](image)
In order to overcome the high simulation cost of crude Monte Carlo, other methods were developed. A first approach were DOE (design of experiments). For this purpose, only few samples of s are simulated, and then a RSM (response surface model) is built using these samples.

Usually only linear models are used for this purpose, because the effort for creating higher order models increases more than linearly with the number of dimensions of s. For example, to determine a quadratic model in a space of p dimensions, you need \((p^2+p)/2+p+1\) simulation runs. For a linear model you need only \(p+1\) simulation runs. For a circuit with 100 transistors with mismatch in \(v_{th}\) and \(u_0\), and 25 process variables, \(p=225\). So you need 226 simulations for the linear model, and 25651 simulations for a quadratic model.

After creating the RSM, one evaluation in the RSM can be calculated with very little effort, therefore millions of Monte Carlo runs are possible in the model. Actually, a Monte Carlo simulation is not even needed anymore, because mean value, standard deviation and yield can be calculated directly from the coefficients of the linear model with highest accuracy.

Figure 2 shows a Monte Carlo simulation in a linear RSM. The number of samples is larger than in Fig. 1, but they are not calculated for the real performance function, but only for the model. The model is created by a DOE around \(s=\mu\) and therefore is most accurate around this point where the sample density is highest.

The disadvantages of this approach are obvious from Figure 2: The model error is large for samples that are far away from \(s=\mu\). But these samples are most important for yield estimation.

RSM in DOEs often gives reasonably accurate results for estimation of the mean value and standard deviation of \(f\), because these statistics depend on the majority of samples.

This method can create large relative errors for yield estimation, because the yield depends on a small number of samples that violates the specification and that are far away from \(s=\mu\). Even performance functions that are nearly linear around \(s=\mu\) and show a high correlation with the linear model can contain small non-linearities further away. Since the „goodness of fit“ \(R^2\) is also dominated by the majority of samples, the \(R^2\) of such a performance can be very close to 1, while nevertheless the relative error of the yield estimate is large.
These drawbacks led to the development of a more advanced method for yield estimation in the 1990s, which is called „worst-case analysis“. The idea of it is, that the model for yield estimation must not be built around s=mu, but around s=swc. The dashed line in Figure 3 shows this model.

Since the value of swc is not known initially, it has to be calculated first. This is similar to an optimization problem: Find the value of s that is closest to mu and for which f(s) violates its specification. It is only mathematically an optimization problem; the circuit design parameters are not modified during worst-case analysis, therefore it's called „analysis“ and not „optimization“.

After calculation of swc, the linear model is calculated around s=swc and yield is estimated. As you can see, the model is not useful for estimation of mean and standard deviation of f, but concentrates its accuracy on the part of the process space that's relevant for yield estimation.

Worst-case analysis consumes more simulation time than a simple DOE, because swc has to be determined first. The advantage is, that yield estimation is much more accurate. Furthermore, knowing the worst-case point swc is necessary for today's most efficient yield optimization algorithms. Yield optimization based on crude Monte Carlo or on DOE+RSM is theoretically possible, but it is very slow and inaccurate.

2. Application in WiCkeD

WiCkeD provides crude Monte Carlo analysis and worst-case analysis. For estimation of mean values and standard deviations, we recommend using a Monte Carlo simulation with a sample size of N=150. This is also sufficient for a quick and rough estimate of the yield and consumes less simulation effort than DOE+RSM in most cases.

If you need more precise information about the yield, like „is it 99.0% or 99.9%“, this is very expensive with Monte Carlo, and simple DOE+RSM is too inaccurate. For this purpose we provide worst-case analysis.

Yield optimization in WiCkeD is based on worst-case analysis.